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Plastic flow in a conical channel Qualitative features of the solutions under different yield conditions☆

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Abstract

The problem of the convergence of the solutions of problems of plasticity theory, with a yield condition which depends on the hydrostatic stress, to solutions based on classical plasticity theory with von Mises or Tresea conditions is considered, with a particular choice of the parameters of the material model. For the case of axisymmetric flow of material in a channel with converging and diverging walls, solutions according to two plasticity theories with a yield condition which depends on the hydrostatic stress are compared with the classical solution. It is shown that only the solution using Spencer's model shows all the main features of the classical solution. As the internal criterion of the choice of the preferred plasticity theory when examining a special class of problems, it is suggested that the criterion of the convergence of the solutions to the solutions of classical plasticity theory should be used.

Theories of rigid–plastic bodies, based on yield conditions which depend on the hydrostatic stress but including the incompressibility equation, are used to describe the motion of granulated and granular materials and the deformation of certain metal alloys.[1–4] In the latter case, the stress states both for negative and positive hydrostatic stresses are of interest.

For the class of plasticity theories considered, the qualitative behaviour of the solutions near surfaces with maximum permissible friction stresses (the maximum friction law) can depend greatly on the constitutive relations. This has been shown[5] for certain semi-analytical solutions obtained within the framework of plane deformation, for the models of Spencer[3] and Hill (for the model equations, see, for example, Ref. 6). An analysis has been made carried out[7] of the solutions obtained[8,9] for the compression of a material, obeying the Spencer and Hill models, between two plates rotating about a common axis, and of the solutions obtained[7] on the assumption that the plates stretch the material. It turned out that the qualitative behaviour of the solutions can be influenced both by the model of the material and by the direction of rotation of the plates. All the solutions examined earlier[7] were obtained within the framework of plane deformaton, and the analysis of the qualitative differences was based on the properties of characteristics of the system of equations. It is therefore of interest to analyse non-planar flows in cases when the system of equations is not of the hyperbolic type. Such conditions are satisfied by the axisymmetric flow of a medium obeying a conical yield condition and the condition proposed by Ishlinskii[1] of coaxiality of the stress and strain rate tensors.

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Thus, the adopted model essentially extends Hill's model, which was proposed for the state of plane deformaton (see Ref. 6), to the case of axisymmetric deformaton. A suitable model problem corresponding to the requirements mentioned above is that of the flow of material through an infinite channel with converging or diverging walls, when the maximum possible friction stresses are acting on the walls. In classical plasticity theory, based on an arbitrary yield condition independent of the hydrostatic pressure, the solution of the corresponding problem is known for converging flow,[10] while for diverging flow it can be obtained in a similar way and differs only in sign. The solution for the converging flow of a material obeying Spencer's model is also known;[3] this solution retains all the main features of the classical solution,[10] including the asymptotic singular behaviour of functions when the friction surface is approached. Solutions have been obtained[11–14] for other models of materials.

1. Statement of the problem

1.1. The General solution

We will consider the problem of the flow of material through an infinite conical channel with converging or diverging walls, on the assumption that the maximum possible friction stresses are acting on the walls. A diagram of the flow in a spherical system of coordinates r, θ, ϕ is shown in Fig. 1, where θ_0 is the flare angle of the conical channel. We will assume[10] that the velocity components $u_\theta = u_\phi = 0$. Then the incompressibility equation gives[10]

$$u_r = \varepsilon Q u(\theta)/r^2 \tag{1.1}$$

where $u(\theta) > 0$ is an arbitrary function, Q is the flow rate of the material and $\varepsilon = \pm 1$, the upper sign corresponding to diverging flow, and the lower sign to converging flow.

From Eq. (1.1) it is possible to find the non-zero components of the strain rate tensor in the form

$$\xi_{rr} = -\frac{2\varepsilon Q u}{r^3}, \quad \xi_{\theta\theta} = \xi_{\phi\phi} = \frac{\varepsilon Q u}{r^3}, \quad \xi_{r\theta} = \frac{\varepsilon Q}{2r^3}\frac{du}{d\theta} \tag{1.2}$$

We will adopt a yield condition of the form

$$\alpha\sigma + \sigma_{eq} = \sigma_0 \tag{1.3}$$

where α and σ_0 are constants of the material, $\sigma = \sigma_{ij}\delta_{ij}/3$ is the hydrostatic stress, $\sigma_{eq} = \sqrt{3/2}(s_{ij}s_{ij})^{1/2}$ is the equivalent stress, σ_{ij} are the components of the stress tensor and $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$. In the space of the principal stresses, condition (1.3) is represented in the form of a conical surface with its apex on the hydrostatic axis. In classical plasticity theory, for the von Mises yield condition, $\alpha = 0$ and σ_0 is the yield point under uniaxial tension. The condition of coaxiality of the stress and strain rate tensors can be written in the form

$$\xi_{ij} = \lambda s_{ij} \tag{1.4}$$

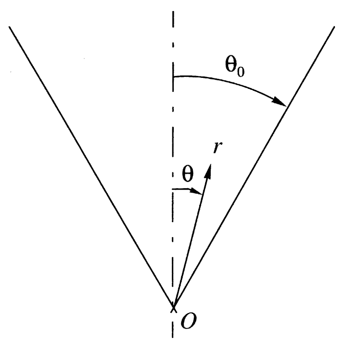

Fig. 1.

where $\lambda > 0$ is a coefficient of proportionality. Taking into account relations (1.2) and the identity

$$s_{rr} + s_{\theta\theta} + s_{\phi\phi} = 0$$

we obtain

$$s_{rr} = -2s_{\theta\theta} = -2s_{\phi\phi} \tag{1.5}$$

Using these equations, and introducing the substitution

$$s_{\theta\theta} = \tau\cos\gamma/3, \quad s_{r\theta} = \tau\sin\gamma/\sqrt{3}, \quad \tau > 0 \tag{1.6}$$

we can write condition (1.3) in the form

$$\alpha\sigma + \tau = \sigma_0 \tag{1.7}$$

Taking into account relations (1.5) and (1.7), we will express the stresses σ_{rr} and $\sigma_{\theta\theta}$ in the form

$$\sigma_{rr} = \sigma - 2\tau\cos\gamma/3, \quad \sigma_{\theta\theta} = \sigma + \tau\cos\gamma/3$$

Furthermore, we will assume that γ is independent of r. The equilibrium equations will then take the form

$$\begin{aligned} &r\left(1 + \frac{2}{3}\alpha\cos\gamma\right)\frac{\partial\sigma}{\partial r} - \frac{\alpha\sin\gamma}{\sqrt{3}}\frac{\partial\sigma}{\partial\theta} + (\sigma_0 - \alpha\sigma)\left(\frac{\cos\gamma}{\sqrt{3}}\frac{d\gamma}{d\theta} - 2\cos\gamma + \frac{\sin\gamma\,\mathrm{ctg}\,\theta}{\sqrt{3}}\right) = 0 \\ &-\frac{\alpha r\sin\gamma}{\sqrt{3}}\frac{\partial\sigma}{\partial r} + \left(1 - \frac{\alpha\cos\gamma}{3}\right)\frac{\partial\sigma}{\partial\theta} + (\sigma_0 - \alpha\sigma)\left(\sqrt{3} - \frac{1}{3}\frac{d\gamma}{d\theta}\right)\sin\gamma = 0 \end{aligned} \tag{1.8}$$

These equations are consisent if

$$\ln(1 - \alpha\sigma/\sigma_0) = A\ln r + p(\theta) \tag{1.9}$$

where A is a constant and $p(\theta)$ is an arbitrary function of θ. Substituting expression (1.9) into Eq. (1.8), we obtain equations for $p(\theta)$ and $\gamma(\theta)$

$$\begin{aligned} &\sqrt{3}\alpha(\alpha - 3\cos\gamma)\frac{d\gamma}{d\theta} = \Phi(\gamma, \theta) \\ &\sqrt{3}(\alpha - 3\cos\gamma)\frac{dp}{d\theta} = \sin\gamma[A(3 - \alpha\cos\gamma) - \sqrt{3}\alpha(\sin\gamma\,\mathrm{ctg}\,\theta + \sqrt{3}\cos\gamma)] \\ &\Phi(\gamma, \theta) = \alpha^2(A + 3)(2 + \sin^2\gamma) - 3\alpha A\cos\gamma - \sqrt{3}\cos\gamma\sin\gamma\,\mathrm{ctg}\,\theta - 9A + \\ &+ 3\sqrt{3}\alpha(\sin\gamma\,\mathrm{ctg}\,\theta - 2\sqrt{3}\cos\gamma) \end{aligned} \tag{1.10}$$

The second equation of this system can be rewritten in the form

$$\frac{dp}{d\gamma} = \frac{\alpha\sin\gamma[A(3 - \alpha\cos\gamma) - \sqrt{3}\alpha(\sin\gamma\,\mathrm{ctg}\,\theta + \sqrt{3}\cos\gamma)]}{\Phi(\gamma, \theta)} \tag{1.11}$$

The coefficient of the derivative in the first equation of system (1.10) vanishes when $\gamma = \gamma_c = \pm\arccos(\alpha/3)$. Suppose $\gamma = \gamma_c$ when $\theta = \theta_c$. Then, in the general case, close to this point, the equation can be represented in the form

$$\pm\sqrt{3}\alpha(9 - \alpha^2)^{1/2}(\gamma - \gamma_c)\frac{d\gamma}{d\theta} = \Phi(\gamma_c, \theta_c) \tag{1.12}$$

where the upper sign corresponds to the condition $\gamma_c > 0$, and the lower sign to the condition $\gamma_c < 0$.

Integrating Eq. (1.12), close to the point $\theta = \theta_c$, we obtain

$$\pm\frac{\sqrt{3}\alpha}{2}(9 - \alpha^2)^{1/2}(\gamma - \gamma_c)^2 = (\theta - \theta_c)\Phi(\gamma_c, \theta_c) \tag{1.13}$$

The left-hand side of this equation does not change sign on passing through the point $\gamma = \gamma_c$, while the right-hand side does change sign on passing through the corresponding point $\theta = \theta_c$. Consequently, a solution exists only for one side of the surface $\theta = \theta_c$, and, if the surface $\theta = \theta_c$ is situated in the plastic zone, then the condition $\theta_c = \theta_0$ should be satisfied.

From relation (1.4) it is possible to obtain

$$\xi_{\theta\theta}/\xi_{r\theta} = s_{\theta\theta}/s_{r\theta} \quad (1.14)$$

Then, from Eqs. (1.2) and (1.6) we have the equation

$$\frac{du}{d\theta} = 2\sqrt{3}u\,\mathrm{tg}\gamma \quad (1.15)$$

in which, changing to differentiation with respect to γ using the first equation of system (1.10), we obtain

$$\frac{du}{d\gamma} = \frac{6\alpha(\alpha - 3\cos\gamma)\,\mathrm{tg}\gamma}{\Phi(\gamma, \theta)}u \quad (1.16)$$

2. Converging flow

In the case of converging flow, the friction stresses are directed away from the point O (Fig. 1). Therefore, from relations (1.6) it follows that $s_{r\theta} > 0$ and $\sin\gamma > 0$. Furthermore, from the second equation of system (1.2) we establish that $\xi_{\theta\theta} < 0$, and then, from relations (1.4) and (1.6), it follows that $s_{\theta\theta} < 0$ and $\cos\gamma < 0$. Thus, the angle γ can vary in the range

$$\pi/2 \le \gamma \le \pi \quad (2.1)$$

The angle γ_c, which in this case is positive, lies outside the limits of the interval (2.1), and therefore the first equation of system (1.10) and then Eq. (1.11) can be solved numerically in the entire interval $0 \le \theta \le \theta_0$.

One of the boundary conditions for the first equation of system (1.10) is the condition on the axis of symmetry

$$\gamma = \pi \quad \text{when} \quad \theta = \theta_0$$

The second condition, which is necessary for determining the constant A, must be formulated on the friction surface $\theta = \theta_0$. Bearing in mind the interval (2.1), the maximum friction law in this case can be formulated as

$$\gamma = \pi/2 \quad \text{when} \quad \theta = \theta_0$$

Since $|tg\gamma| \to \infty$ when $\gamma \to \pi/2$, it follows that, near the friction surface, the solution of Eq. (1.16) has the form

$$u = u_0(\gamma - \pi/2)^{-B}, \quad B = 6\alpha^2/\Phi(\pi/2, \theta_0) \quad (2.2)$$

where u_0 is the constant of integration. Since infinite velocities have no physical meaning, the inequality $\Phi(\pi/2, \theta_0) < 0$ must be satisfied. This condition must be ensured by the magnitude of A. Then, from expression (2.2) it follows that $u=0$ on the friction surface, i.e. sticking occurs. Nonetheless, the magnitude of the derivative $du/d\theta$ can approach infinity when $\theta \to \theta_0$. Since the derivative $d\gamma/d\theta$ is bounded, from solution (2.2) it follows that $|du/d\theta| \to \infty$ when $\theta \to \theta_0$ if

$$1 + B > 0 \quad (2.3)$$

The relation $u(\gamma)$ can be determined by numerical integration. From Eq. (1.16) we obtain

$$u = c\nu(\pi, \gamma), \quad \nu(\beta, \gamma) = \exp\left[6\alpha\int_\beta^\gamma \frac{(\alpha - 3\cos\vartheta)\,\mathrm{tg}\vartheta}{\Phi(\vartheta, \theta)}d\vartheta\right] \quad (2.4)$$

where c is the constant of integration, while θ in the integrand is a known function of ϑ, owing to the solution of the first equation of system (1.10). Bearing in mind relations (1.1), (1.10) and (2.4), and also the range of variation of γ,

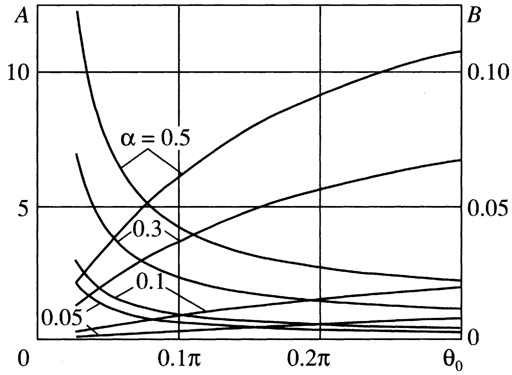

Fig. 2.

for the determination of c we obtain the equation

$$1 = 2\sqrt{3}\pi c\alpha \int_{\pi}^{\pi/2} \frac{(\alpha - 3\cos\gamma)\sin\theta}{\Phi(\gamma, \theta)} \nu(\pi, \gamma) d\gamma \tag{2.5}$$

where, in the integrand, θ is a known function of γ from the solution of the first equation of system (1.10).

The numerical solution of the first equation of system (1.10) under the conditions $\gamma = \pi$ when $\theta = 0$ and $\gamma = \pi/2$ when $\theta = \theta_0$ determines the magnitude of A, the dependence of which on θ_0 is shown in Fig. 2 for certain values of α (the descending curve). Here also we show the dependence on θ_0 of the quantity B, which defines the behaviour of the velocity near the friction surface in accordance with expressions (2.2). It can be seen that inequality (2.3) is satisfied for all the cases considered. Consequently, in the solution obtained, $|du/d\theta| \to \infty$ when $\theta \to \theta_0$. Knowing the relation $\theta(\gamma)$, it is possible to find the quantity c from Eq. (2.5), and then $u(\gamma)$ from relation (2.4). Thus, we obtain the relation $u(\theta)$, which determines the velocity profile in accordance with Eq. (1.1) in parametric form. This relation is shown in Fig. 3 for some values of α when $\theta_0 = 30°$. It can be seen that, in a very narrow region near the friction surface, high velocity gradients arise. It is interesting to note that this narrow region arises not only on the scale of the characteristic size of the process but also in the zone where asymptotic representation (2.2) is used. In the numerical solution, representation (2.2) was used in the range $\pi/2 \le \gamma \le \pi/2 + 0.001$. Fig. 3 also shows the behaviour of the function $u(\gamma)$ in this range when $\theta_0 = 30°$.

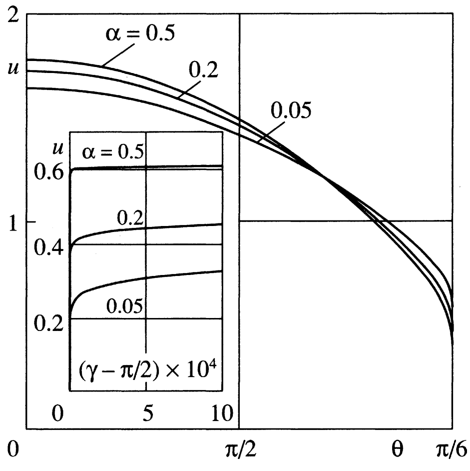

Fig. 3.

3. Diverging flow

In the case of diverging flow, the friction stresses are directed towards the point O (Fig. 1), and therefore from relations (1.6) it follows that $s_{r\theta} < 0$ and $\sin\gamma < 0$. Furthermore, from the second equation of (1.2) we establish that $\xi_{\theta\theta} > 0$, and from relations (1.4) and (1.6) it then follows that $s_{\theta\theta} > 0$ and $\cos\gamma > 0$. Thus, the angle γ can vary in the range

$$-\pi/2 \le \gamma \le 0 \tag{3.1}$$

The angle γ_c, which in this case is negative, lies within this range, and, as follows from Eq. (1.13), the maximum possible friction stresses arise if $\gamma = \gamma_c$ and $\theta = \theta_0 = \theta_c$. This condition is the maximum friction law in the case under examination. The second boundary condition is the condition on the axis of symmetry

$$\gamma = 0 \quad \text{when} \quad \theta = 0$$

Taking Eq. (1.12) into account, Eq. (1.16) near the friction surface can be rewritten in the form

$$\frac{du}{u} = -\frac{6\alpha \operatorname{tg}\gamma_c(9-\alpha^2)}{\Phi(\gamma_c, \theta_0)}(\gamma - \gamma_c)d\gamma$$

and after integration we obtain

$$u = u_1 \exp[\Lambda_0(\gamma-\gamma_c)^2], \quad \Lambda_0 = -\frac{3\alpha \operatorname{tg}\gamma_c\sqrt{9-\alpha^2}}{\Phi(\gamma_c, \theta_0)} \tag{3.2}$$

where u_1 is the constant of integration. From relation (3.2) it follows that on the friction surface $u = u_1 \neq 0$, and thus slip occurs on this surface. Furthermore, since the angle γ varies in the range $-\pi/2 < \gamma_c \le \gamma \le 0$, it follows from Eq. (1.15) that the shear strain rate is bounded everwhere.

Assuming that close to the axis of symmetry

$$\gamma = K\theta + o(\theta), \quad \theta \to 0 \tag{3.3}$$

and substituting this expression into the first equation of system (1.10), we obtain

$$K = \frac{(\alpha-3)[6\alpha + A(2\alpha+3)]}{\sqrt{3}(\alpha^2 - 6\alpha + 1)} \tag{3.4}$$

The ranges of variation of γ and θ assume that $K < 0$. For the actual values of α, the inequality $\alpha < 3$ is satisfied, and therefore it follows from Eq. (3.4) that A must satisfy the conditions

$$A < -\frac{6\alpha}{2\alpha+3} \quad \text{when} \quad \alpha > \alpha_c = 3 - 2\sqrt{2} \approx 0.1715, \quad A > -\frac{6\alpha}{2\alpha+3} \quad \text{when} \quad \alpha < \alpha_c \tag{3.5}$$

We will assume that the solution of the first equation of system (1.10) exists over the entire range $0 \le \theta \le \theta_0$ and accordingly in the range $-\gamma_c \le \gamma \le 0$. For this, it is necessary for the function $\Phi(\gamma, \theta)$ not to vanish at any point of the corresponding open ranges. In particular, one of the necessary critical conditions has the form $\Phi(\gamma_c, \theta_0) = 0$. From this it is possible to find the critical value of A. A solution can exist provided

$$A < A_{cr}, \quad A_{cr} = -\left[\frac{8\sqrt{3}\alpha \operatorname{ctg}\theta_0}{(9-\alpha^2)^{1/2}} - 3\alpha^2\right]\frac{1}{(9-\alpha^2)} \tag{3.6}$$

From the first inequality of (3.5) and (3.6) it can be seen that, when $\alpha > \alpha_c$, the range of permissible values of A is unlimited, while from the second inequality of (3.5) and (3.6) it follows that, when $\alpha < \alpha_c$, this range is limited and may even be empty. For $\alpha > \alpha_c$, the solution of the first equation of system (1.10) under the formulated boundary conditions may be obtained numerically without difficulties. This solution is shown in Fig. 4. For $\alpha < \alpha_c$, the structure of the solution is much more complex. A solution may not exist, and, when it does, it may be non-unique.

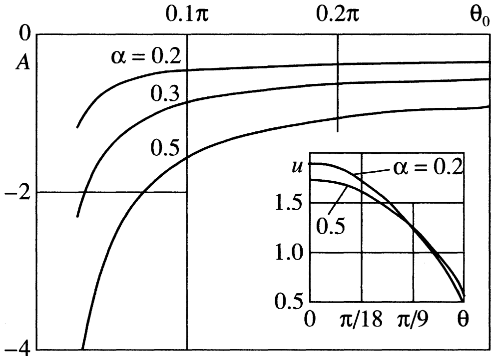

Fig. 4.

For a more detailed description of the structure of the solution in this region, additional investigations are needed. If a solution exists, the velocity field satisfies Eq. (3.2) near the friction surface, which is of greatest interest for the purposes of the present paper.

The solution of Eq. (1.16) can be written in the form

$$u = c_1 \exp\left[6\alpha\int_0^{\gamma}\frac{(\alpha-3\cos\vartheta)\,\mathrm{tg}\,\vartheta}{\Phi(\vartheta,\theta)}d\vartheta\right] \tag{3.7}$$

where c_1 is the constant of integration, and θ in the integrand is a known function of ϑ, owing to the solution of the first equation of system (1.10). Bearing in mind relations (1.1), (1.10) and (3.7), and also the range of variation of γ, for the determination of c_1 we obtain the equation

$$1 = 2\sqrt{3}\pi c_1\alpha\int_0^{\gamma_c}\frac{(\alpha-3\cos\gamma)\sin\theta}{\Phi(\gamma,\theta)}\exp\left[6\alpha\int_0^{\gamma}\frac{(\alpha-3\cos\vartheta)\,\mathrm{tg}\,\vartheta}{\Phi(\vartheta,\theta)}d\vartheta\right]d\gamma \tag{3.8}$$

In the factor in front of the exponent, θ is a known function of γ from the solution of the first equation of system (1.10).

Since the ralation θ(γ) was defined earlier, from Eq. (3.8) it is possible to find the value of c_1, and then the value of $u(\gamma)$ from Eq. (3.7). In this way, we obtain the relation $u(\theta)$, which determines the velocity profile, in accordance with Eq. (1.1), in parametric form. This relation is shown in Fig. 4 for some values of α when $\theta_0 = 30°$.

4. Comparison of the solutions, and conclusion

To complete the solution, it is necessary to integrate Eq. (1.11) for both types of flow. However, since the quantity *A* and the function γ(θ) have been determined, this integration presents no difficulties. Moreover, the stress distribution is not essential for the purposes of the present investigation, and we will therefore focus below chiefly upon velocity field analysis.

Comparing the nature of the velocity field behaviour in the vicinity of the friction surface in the case of converging and diverging flows, as follows from representations (2.2) and (3.2), it can be seen that the solutions differ qualitatively and correspond to different friction regimes (stick or slip). It is of interest to classify the known solutions, based on other models of materials, in terms of this feature. These solutions were obtained for converging flow. However, for materials with a yield condition not depending on the hydrostatic stress, and for materials obeying Spencer's model,[3] there is no fundamental difference between converging and diverging flows. In solutions for rigid–ideally plastic material[10] and for Spencer's model,[3] a slipping regime arises. In solutions for a viscoplastic material[11] and rigid–plastic hardening material,[12] a sticking regime should have arisen in accordance with the general theory.[15–17]. However, a direct check ascertains that the solutions in Refs. 11,12 do not exist by the maximum friction law, since the structure of these solutions does not enable the sticking condition to be satisfied. Nevertheless, for the creep theory model, which may be regarded as a special case of a viscoplasticity model, a solution under the sticking condition exists and was obtained earlier.[13]

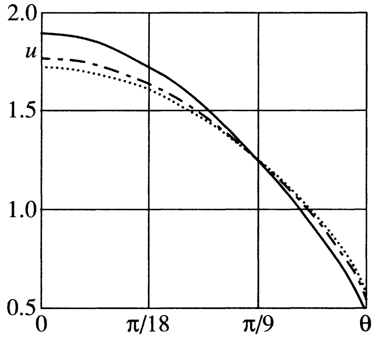

Fig. 5.

The solution obtained for converging flow satisfies inequality (2.3), so that $|du/d\theta| \to \infty$ as the friction surface is approached. Here, the quantity B depends on the process parameters and the properties of the material (Fig. 2). The velocity fields obtained earlier,[3,10] are again such that $|du/d\theta| \to \infty$ as the friction surface is approached. However, in these solutions, the first singular term is always of the order of $1/\sqrt{s}$, where s is the distance to the friction surface. Moreover, such behaviour of the velocity field is a feature of the corresponding models[18–20] and not of the specific solution. The solution obtained for diverging flow leads to a velocity field with a bounded derivative $du/d\theta$. In this sense, the given solution corresponds more to the well-known solution.[13] However, its structure depends greatly on the quantity α and, in particular, on the relation between α and α_c (in exactly the same way as the flow of a two-layer material, on the assumption that each layer obeys the model of a rigid–ideally plastic material.[14]). For $\theta_0 = 30°$ and $\alpha = 0.2$, the function $u(\theta)$ is shown in Fig. 5 for diverging flow (the dashed curve), for converging flow (the dotted curve) and for the flow of a rigid–ideally plastic material[10] (the dot-dash curve). It can be seen that the radial velocity distributions differ little at all points of the deformed region, with the exception of a very narrow zone close to the friction surface. It seems that such a significant difference in behaviour of the solutions for similar boundary-value problem formulations may be important in developing numerical methods for analysing more complex processes. Furthermore, the velocity field largely determines the change in properties of the material during deformation. We know from experimental investigations that the properties of metals in a narrow region close to the friction surface differ considerably from the properties in the bulk of the material.[21,22] Taking into account the large variety of models of plastic materials with the yield condition which depends on the hydrostatic stress,[3,6] a comparison of these experimental results with the obtained behaviour of solutions may serve as one of the criteria for selecting a suitable model of the material.

The stress field in a solution based on the model of a rigid–ideally plastic material does not satisfy the necessary condition $\sigma_{\theta\theta} < 0$ at all points of the friction surface. In particular, this has been pointed out earlier for plane flow.[23,24] However, using a constant of integration, the range in which the noted condition is satisfied can be made as wide as desired. The solution obtained also has this drawback, and the corresponding constant appears in the integration of Eq. (1.11). Using relations (1.6), (1.7) and (1.9), we obtain

$$\frac{\sigma_{\theta\theta}}{\sigma_0} = \frac{\cos\gamma}{3} + (1 - r^A e^p)\left(\frac{1}{\alpha} - \frac{\cos\gamma}{3}\right) \tag{3.9}$$

For converging flow, $\gamma = \pi/2$ on the friction surface and $A > 0$ (Fig. 2), and therefore it follows from Eq. (3.9) that there is always a region $0 \le r < r_0$ on the friction surface where $\sigma_{\theta\theta} > 0$. However, the value of r_0 can be made as small as desired. For diverging flow, $\gamma = \gamma_c$ on the friction surface and $A < 0$ (Fig. 4), and therefore it follows from relation (3.9) that there is always a region $R_0 < r$ on the friction surface where $\sigma_{\theta\theta} > 0$. However, the value of R_0 can be made as large as desired. This drawback of the solution has no effect on the general conclusions relating to the local behaviour of solutions near the friction surface.

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